The Asymptotic of Transmission Eigenvalues for a Domain with a Thin Coating

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THE ASYMPTOTIC OF TRANSMISSION EIGENVALUES FOR A DOMAIN WITH A THIN COATING

H. BOUJLIDA *, H. HADDAR†, AND M. KHENISSI ‡

Abstract. We consider the transmission eigenvalue problem for a medium surrounded by a thin layer of inhomogeneous material with different refractive index. We derive explicit asymptotic expansion for the transmission eigenvalues with respect to the thickness of the thin layer. We prove error estimate for the asymptotic expansion up to order 1. This expansion can be used to obtain explicit expressions for constant index of refraction.

Key words. transmission eigenvalues, asymptotic expansions, thin layers, inverse scattering problems

AMS subject classifications. 35P30, 35P25

1. Introduction.

This work is a contribution to the study of transmission eigenvalues [11, 4, 6] and their relation to the shape and material properties of scatterers. These special frequencies are associated with the existence of an incident field that does not scatter. They can be equivalently defined as the eigenvalues of a system of two coupled partial differential equations posed on the inclusion domain. One of these equations refers to the equation satisfied by the total field and the other one is satisfied by the incident field. The two equations are coupled on the boundary by imposing that the Cauchy data coincide. This eigenvalue problem can then be formulated as an eigenvalue problem for a non-selfadjoint compact operator. Although non intuitive, it can be shown that this problem admits an infinite discrete set of real eigenvalues without finite accumulation points [7, 26]. These special frequencies can be identified from far field data as proved in [5, 19, 4]. Since they carry information on the material properties of the scatterer, transmission eigenvalues would then be of interest for the inverse problem of retrieving qualitative information on the material properties from measured multistatic data [14, 15]. In this perspective, it appears important to study the dependence of these eigenfrequencies with respect to the material properties and the geometry. Several works in the literature have addressed this issue by considering asymptotic regimes and quantifying the dependence of the first leading terms in the asymptotic expansion of the transmission eigenvalue with respect to the small parameter [10, 8, 21, 16]. We here consider the case of a scatterer made of a thin coating which corresponds to frequently encountered configurations in the stealth technology for instance. The goal is to characterize the dependence of the first order term on the material properties and the thickness of the coating. A first work on this topic was done in [10] where the case of coated perfect scatterer is considered. One proves in particular for the latter case that the first order term depends only on the thickness. We here address the more complicated configuration of a coated penetrable media. The analysis indicates that the first order asymptotic resembles to the shape derivative for the buckling plate equation [17] and contain non trivial dependence on the material properties. More importantly, this expansion allows us to obtain explicit (approximate) expressions for the thin layer index of refraction in terms of the thickness of the layer, the transmission eigenvalue for the coated medium that can be extracted from the measurements and the transmission eigenvalues and eigenvectors for the coated free medium that can be evaluated numerically. This indeed can be useful for the solution of the inverse problem.

Although the formal derivation follows the systematic procedure using the classical scaled expansion method (as in [3, 2, 13] for instance), the rigorous justification is much more involved. For instance the arguments in [10] are hard to extend to the present case since special uniform estimates have to be obtained for the transmission problem. We restrict ourselves here to the justification of the first two terms in the asymptotic expansion using the abstract theory developed in [23, 21]. We follow the procedure developed in [8] for the case of small obstacles asymptotic. The main technical point in the proof is to obtain the corrector for the main operator, which is

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here the biharmonic operator. Our main result provides explicit expansion for simple transmission eigenvalues and for multiple transmission eigenvalues that are associated with a generalized eigenspace spanned only by eigenvectors.

We analyze the problem where the contrast in material properties affect only the lower order term in the Helmholtz equation. We finally indicate that although the problem is considered only in dimension 2, the results of the main theorem (including the expression of the first order asymptotic term) remain true for three dimensions (up to more complicated technicalities in the proof related to differential geometry).

The paper is organized as follows. We first introduce the transmission eigenvalues and write them as the eigenvalues of a non selfadjoint operator. We then present the main result of our paper and discuss applications to the inverse problem. We present next the outline of a classical formal procedure to obtain the expression of the asymptotic expansion. We give the expression till the second order term. We explain in particular why the expression of the second order term would have less interest in practice. We then proceed with the main part of the paper that provides explicit expressions and an error estimate for the first two terms in the asymptotic expansion.

2. Problem statement and main results. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ. We denote by

$$\Omega_\epsilon^0 = \{x \in \Omega, \quad d(x,\Gamma) > \epsilon\}$$

and its boundary

$$\Gamma_\epsilon = \{x \in \Omega, \quad d(x,\Gamma) = \epsilon\} = \partial\Omega_\epsilon^0,$$

for $\epsilon > 0$ a small enough parameter, where $d(x,\Gamma)$ denotes the distance of a point x to the boundary Γ. Let $\Omega_\epsilon = \Omega \backslash \overline{\Omega_\epsilon^0}$ be the layer of thickness ϵ around Ω_ϵ^0 (see Figure 1).

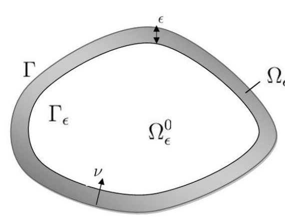

FIG. 1. *Stretch of the geometry*

We consider the following transmission eigenvalue problem:

$$(1) \qquad \begin{cases} \Delta w_\epsilon + k_\epsilon^2 n_\epsilon(x) w_\epsilon = 0 & \text{in} \quad \Omega, \\ \Delta v_\epsilon + k_\epsilon^2 v_\epsilon = 0 & \text{in} \quad \Omega, \\ w_\epsilon = v_\epsilon & \text{on} \quad \Gamma, \\ \dfrac{\partial w_\epsilon}{\partial \nu} = \dfrac{\partial v_\epsilon}{\partial \nu} & \text{on} \quad \Gamma, \end{cases}$$

where k_ϵ denotes the unknown eigenfrequency and ν the unitary normal to Γ directed to the interior of Ω. The index of refraction n_ϵ is defined as follows:

$$n_\epsilon(x) = \begin{cases} n_0(x) & \text{in } \Omega_\epsilon^0, \\ n_1(x) & \text{in } \Omega_\epsilon, \end{cases}$$

where n_0 and n_1 are non negative real valued functions $\in L^\infty(\mathbb{R}^2)$ that are independent from ϵ. For the sake of simplicity, we assume that the restriction of n_0 and n_1 to Ω_ϵ are constant functions along the normal coordinate to Γ for ϵ sufficiently small. We finally assume that the function $1/(1-n_\epsilon)$ is either positive definite or negative definite on Ω. We remark that this assumption also implies that $1/(1-n_0)$ is either positive definite or negative definite on Ω and that

$$(2) \qquad 1/|1-n_\epsilon(x)| \geq \gamma > 0 \quad \text{for a.e. } x \in \Omega$$

with γ being independent from (sufficiently small) ϵ.

The main goal of this paper is to find the asymptotic expansion of eigenfrequencies k_ϵ with respect to ϵ. Assuming that $\dfrac{1}{1-n_\epsilon} \in L^\infty(\Omega)$, the transmission eigenvalue problem (1) can be reformulated as the nonlinear eigenvalue problem for $\lambda_\epsilon := k_\epsilon^2 \in \mathbb{R}$ and $u_\epsilon := w_\epsilon - v_\epsilon \in H_0^2(\Omega)$ such that

$$(\Delta + \lambda_\epsilon n_\epsilon)\frac{1}{1-n_\epsilon}(\Delta + \lambda_\epsilon)u_\epsilon = 0 \quad \text{in } \Omega,$$

which in variational form, after integration by parts, is formulated as finding $\lambda_\epsilon \in \mathbb{R}$ and non-trivial function $u_\epsilon \in H_0^2(\Omega)$ such that

$$\int_\Omega \frac{1}{1-n_\epsilon}(\Delta u_\epsilon + \lambda_\epsilon u_\epsilon)(\Delta\phi + \lambda_\epsilon n_\epsilon \phi)dx = 0, \quad \forall \phi \in H_0^2(\Omega). \tag{3}$$

The space $H_0^2(\Omega)$ denotes the closure in $H^2(\Omega)$ of the set of regular compactly supported functions in Ω. We shall work with the reformulation of (3) as a linear eigenvalue problem for a non selfadjoint compact operator [4]. First observe that (3) can be written as

$$A_\epsilon u_\epsilon + \lambda_\epsilon B_\epsilon u_\epsilon + \lambda_\epsilon^2 C_\epsilon u_\epsilon = 0 \ \text{ in } \ H_0^2(\Omega) \tag{4}$$

where

$$A_\epsilon : H_0^2(\Omega) \to H_0^2(\Omega), \quad B_\epsilon : H_0^2(\Omega) \to H_0^2(\Omega), \quad C_\epsilon : H_0^2(\Omega) \to H_0^2(\Omega)$$

are defined by the Riesz representation theorem as

$$(A_\epsilon u_\epsilon, \phi)_{H_0^2(\Omega)} := \int_\Omega \frac{1}{1-n_\epsilon}\Delta u_\epsilon \Delta\phi dx, \tag{5}$$

$$(B_\epsilon u_\epsilon, \phi)_{H_0^2(\Omega)} := \int_\Omega \frac{1}{1-n_\epsilon}(u_\epsilon \Delta\phi + n_\epsilon \Delta u_\epsilon \phi)dx, \tag{6}$$

and

$$(C_\epsilon u_\epsilon, \phi)_{H_0^2(\Omega)} := \int_\Omega \frac{n_\epsilon}{1-n_\epsilon}u_\epsilon \phi dx. \tag{7}$$

Note that $A_\epsilon : H_0^2(\Omega) \to H_0^2(\Omega)$ is a bounded, self-adjoint and invertible operator (thanks to (2)), $B_\epsilon : H_0^2(\Omega) \to H_0^2(\Omega)$ is a bounded, compact and self-adjoint operator and $C_\epsilon : H_0^2(\Omega) \to H_0^2(\Omega)$ is a (non negative or non positive) bounded, compact and self-adjoint operator. Observe that since A_ϵ is invertible, $\lambda_\epsilon \neq 0$. In order to avoid distinguishing the cases of $1 - n_\epsilon$ being positive or negative we shall abusively set $C_\epsilon^{\frac{1}{2}} \equiv -(-C_\epsilon^{\frac{1}{2}})$ in the case where $1 - n_\epsilon$ non positive.

Setting $U_\epsilon = (u_\epsilon, \lambda_\epsilon C_\epsilon^{\frac{1}{2}} u_\epsilon)$, the transmission eigenvalue problem (4) can be transformed into the linear eigenvalue problem, $\tau_\epsilon \in \mathbb{R}$, $U_\epsilon \in H_0^2(\Omega) \times H_0^2(\Omega)$ such that

$$(\mathcal{T}_\epsilon - \tau_\epsilon I)U_\epsilon = 0, \quad \text{with} \quad \tau_\epsilon = \frac{1}{\lambda_\epsilon}, \tag{8}$$

for the compact non-selfadjoint operator $\mathcal{T}_\epsilon : H_0^2(\Omega) \times H_0^2(\Omega) \to H_0^2(\Omega) \times H_0^2(\Omega)$ defined by

$$\mathcal{T}_\epsilon = \begin{pmatrix} -A_\epsilon^{-1}B_\epsilon & -A_\epsilon^{-1}C_\epsilon^{\frac{1}{2}} \\ C_\epsilon^{\frac{1}{2}} & 0 \end{pmatrix}. \tag{9}$$

We set

$$\mathcal{T}_0 = \begin{pmatrix} -A_0^{-1}B_0 & -A_0^{-1}C_0^{\frac{1}{2}} \\ C_0^{\frac{1}{2}} & 0 \end{pmatrix} \tag{10}$$

where A_0, B_0 and C_0 are defined by (5), (6) and (7) respectively for $n_\epsilon = n_0$ in Ω. We state here the main result of this paper which will be proven in Section 4. In the following a transmission eigenvalue λ_0 is called simple if the corresponding $\tau_0 = 1/\lambda_0$ has an algebraic multiplicity equal to 1. We refer to Theorem 4.11 for the case where λ_0 has an associated eigenspace formed only by eigenvectors (and therefore an algebraic multiplicity that coincides with the geometrical multiplicity).

THEOREM 2.1. *Assume that $n_0, n_1 \in C^4(\overline{\Omega})$. Let $\lambda_0 \in \mathbb{R}$ be a simple transmission eigenvalue of (3) with $n_\epsilon = n_0$ in Ω and let $u_0 \in H_0^2(\Omega)$ be an associated eigenfunction. This implies in particular that*

$$\beta_0 := \int_\Omega \frac{1}{1-n_0}\Big(\lambda_0^2 n_0 |u_0|^2 - |\Delta u_0|^2\Big) dx \neq 0.$$

If we suppose in addition that u_0 and $A_0^{-1}u_0$ are in $C^6(\overline{\Omega})$, then, for sufficiently small $\epsilon > 0$, there exists a transmission eigenvalue λ_ϵ of (3) such that

$$\lambda_\epsilon = \lambda_0 + \epsilon\lambda_1 + O(\epsilon^{\frac{3}{2}})$$

where λ_1 is given by the following expression

$$\lambda_1 := \frac{\lambda_0}{\beta_0}\int_\Gamma \frac{n_0 - n_1}{(1-n_0)^2}|\Delta u_0|^2 ds(x).$$

This theorem is an immediate consequence of Theorem 4.8 that is stated and proven in the last section of this paper.

The formal calculations in Section 3 show that the formula for λ_1 is generically valid whenever $\beta_0 \neq 0$. However, we remark that in the case of transmission eigenvalues with multiplicity greater than 1, this is not automatically ensured (See Theorem 4.11 for a rigorous expression of λ_1 that involves all eigenvectors associated with λ_0).

From the practical point of view, this theorem implies in particular that λ_1 gives a measure for the contrast $n_0 - n_1$. For instance, if n_1 is constant and n_0 is constant on Γ, one can approximate the value of n_1 using the identity

$$n_1|_\Gamma = n_0|_\Gamma - \frac{\lambda_\epsilon - \lambda_0}{\epsilon\alpha_0}\int_\Omega \frac{1}{1-n_0}\Big(\lambda_0 n_0|u_0|^2 - \frac{1}{\lambda_0}|\Delta u_0|^2\Big)dx + O(\epsilon^{\frac{1}{2}}) \tag{11}$$

with

$$\alpha_0 := \int_\Gamma \frac{|\Delta u_0|^2}{(1-n_0)^2} ds(x).$$

For the inverse problem where one would like to determine n_1 from multistatic measurements of scattered waves, the value of λ_ϵ can be approximated using sampling methods as in [5, 4] (see also [19] for an alternative approach). The values of λ_0 and u_0 can be computed numerically if one has a priori knowledge of n_0 and Ω (see for instance [12, 18, 20] for numerical methods to approximate λ_0 and u_0). We finally indicate that, although not carefully checked, we conjecture that the expression for λ_1 remains true in three dimensions (corrections due to the curvature of Γ only affect higher order terms).

3. Formal asymptotic expansion. In this section, we derive the formal asymptotic expansion for transmission eigenvalues and give explicit formulas for the terms up to order 2. The idea here is to provide a systematic formal way to quickly obtain the explicit expression of λ_1 in Theorem 2.1 and also higher order terms. The latter turn out to have complicated expressions that would be of marginal interest for the solution of the inverse problem mentioned above. This formal stage will also be helpful in establishing the rigorous proof based of Osborn's theorem [23]. It allows one to have an intuition for the expression of the corrector in the asymptotic of the main operator A_ϵ.

We assume the following expansions for the transmission eigenvalues :

$$\lambda_\epsilon = \sum_{j=0}^{\infty} \epsilon^j \lambda_j, \tag{12}$$

and then follow a classical technique for thin layers asymptotics based on rescaling and asymptotic expansion with respect to the thickness ϵ. We shall mainly follow the approach in [10].

3.1. Scaling. We assume that the boundary Γ is C^∞-smooth, although much less regularity is needed if we restrict ourselves to only few terms in the expansion. The issue of optimal regularity assumptions for Γ is not discussed here. However, one can check that at least a C^2 regularity is needed to get the expression of λ_1. We parametrize Γ as

$$\Gamma = \{x_\Gamma(s), s \in [0, L[\},$$

with L being the length of Γ and s is the curvilinear abscissa. At the point $x_\Gamma(s)$, the unit tangent vector is $\tau(s) := \dfrac{dx_\Gamma(s)}{ds}$, the curvature $\kappa(s)$ is defined by:

$$\frac{d\tau(s)}{ds} = -\kappa(s)\nu(s) \quad \text{or} \quad \frac{d\nu(s)}{ds} = \kappa(s)\tau(s).$$

Within these notations, the boundary of Ω_ϵ^0 is parametrized as

$$\Gamma_\epsilon = \{x_\Gamma(s) + \epsilon\nu(s), s \in [0, L[\}.$$

This parametrization of the surface Γ_ϵ is equivalent to the definition of Γ_ϵ, for $\epsilon > 0$ a small enough parameter.

For a function u defined in Ω_ϵ, we consider $\tilde{u}$ defined on $[0, L[\times]0, \epsilon[$ by

$$\tilde{u}(s,\eta) = u(\varphi(s,\eta)) \text{ where } \varphi(s,\eta) := x_\Gamma(s) + \eta\nu(s). \tag{13}$$

Then, the gradient and Laplace operators are expressed in the local coordinates as:

$$\nabla u = \Big(\frac{1}{(1+\eta\kappa(s))}\frac{\partial}{\partial s}\tau(s) + \frac{\partial}{\partial \eta}\nu(s)\Big)\tilde{u},$$

$$\Delta u = \Big(\frac{1}{(1+\eta\kappa)}\frac{\partial}{\partial s}\frac{1}{(1+\eta\kappa)}\frac{\partial}{\partial s} + \frac{\kappa}{(1+\eta\kappa)}\frac{\partial}{\partial \eta} + \frac{\partial^2}{\partial \eta^2}\Big)\tilde{u}. \tag{14}$$

To make the formal calculations, we need to separate the thin layer and scaled it with respect to the thickness so that the equation are posed on a domain independent from ϵ. We therefore rewrite the transmission eigenvalue problem (1) in the following equivalent form

$$\begin{cases} \Delta w_\epsilon^+ + k_\epsilon^2 n_1 w_\epsilon^+ = 0 & \text{in} & \Omega_\epsilon, \\ \Delta w_\epsilon^- + k_\epsilon^2 n_0 w_\epsilon^- = 0 & \text{in} & \Omega_\epsilon^0, \\ \Delta v_\epsilon + k_\epsilon^2 v_\epsilon = 0 & \text{in} & \Omega, \\ w_\epsilon^+ = w_\epsilon^-, \quad \dfrac{\partial w_\epsilon^+}{\partial \nu} = \dfrac{\partial w_\epsilon^-}{\partial \nu} & \text{on} & \Gamma_\epsilon, \\ w_\epsilon^+ = v_\epsilon & \text{on} & \Gamma, \\ \dfrac{\partial w_\epsilon^+}{\partial \nu} = \dfrac{\partial v_\epsilon}{\partial \nu} & \text{on} & \Gamma. \end{cases} \tag{15}$$

We denote by $\xi = \dfrac{\eta}{\epsilon}$ the stretched normal variable inside Ω_ϵ and define

$$\begin{array}{cccc} \varphi_\epsilon & : & \mathcal{G} = [0, L[\times]0, 1[& \to & \Omega_\epsilon \\ & & (s,\xi) & \mapsto & \varphi_\epsilon(s,\xi) = x_\Gamma(s) + \epsilon\xi\nu(s). \end{array}$$

Then the expression of the Laplace operator in the scaled layer is:

$$\Delta u = \Big(\frac{1}{(1+\xi\epsilon\kappa)}\frac{\partial}{\partial s}\frac{1}{(1+\xi\epsilon\kappa)}\frac{\partial}{\partial s} + \frac{\kappa}{(1+\xi\epsilon\kappa)}\frac{\partial}{\partial \xi} + \frac{1}{\epsilon^2}\frac{\partial^2}{\partial \xi^2}\Big)\hat{u} =: \Delta_{s,\xi}\hat{u} \tag{16}$$

for $\hat{u}(s,\xi) := u(\varphi_\epsilon(s,\xi))$.

The next step is to write the equation for w_ϵ^+ in the scaled domain and solve for the asymptotic expansion of w_ϵ^+ in terms of the boundary values on Γ. These boundary values are given by the asymptotic expansion of v_ϵ. More specifically, setting $\hat{w}_\epsilon(s,\xi) := w_\epsilon^+(\varphi_\epsilon(s,\xi))$, we have that

$$\Delta_{s,\xi}\hat{w}_\epsilon + \lambda_\epsilon n_1 \hat{w}_\epsilon = 0 \text{ in } \mathcal{G} \tag{17}$$

together with the boundary conditions

$$\begin{cases} \hat{w}_\epsilon(s,0) = v_\epsilon(x_\Gamma(s)) & s \in [0, L[, \\ \dfrac{1}{\epsilon}\dfrac{\partial \hat{w}_\epsilon}{\partial \xi}(s,0) = \dfrac{\partial v_\epsilon}{\partial \nu}(x_\Gamma(s)) & s \in [0, L[. \end{cases} \tag{18}$$

We assume that

$$\hat{w}_\epsilon(s,\xi) = \sum_{j=0}^{\infty} \epsilon^j \hat{w}_j(s,\xi), \quad (s,\xi) \in \mathcal{G} \quad \text{and} \quad v_\epsilon(x) = \sum_{j=0}^{\infty} \epsilon^j v_j(x), \quad x \in \Omega \tag{19}$$

for some functions $\hat{w}_j$ defined on $\mathcal{G}$ and v_j defined on Ω that are independent from ϵ. Multiplying (17) by $\epsilon^2(1+\xi\epsilon\kappa)^3$ and using (12), we obtain

$$\sum_{k=0}^{5} \epsilon^k A_k \hat{w}_\epsilon = 0,$$

where $(A_k)_{k=0\ldots5}$ are differential operators of order 2 at maximum with the following expressions for the first fourth terms:

$$\begin{aligned} A_0 =& \frac{\partial^2}{\partial \xi^2}, \\ A_1 =& 3\xi\kappa\frac{\partial^2}{\partial \xi^2} + \kappa\frac{\partial}{\partial \xi}, \\ A_2 =& \frac{\partial^2}{\partial s^2} + 3\xi^2\kappa^2\frac{\partial^2}{\partial \xi^2} + 2\xi\kappa^2\frac{\partial}{\partial \xi} + \lambda_0 n_1, \\ A_3 =& \xi^3\kappa^3\frac{\partial^2}{\partial \xi^2} + \xi^2\kappa^3\frac{\partial}{\partial \xi} - \xi\frac{\partial \kappa}{\partial s}\frac{\partial}{\partial s} + \xi\kappa\frac{\partial^2}{\partial s^2} + 3\lambda_0 n_1 \xi\kappa + \lambda_1 n_1. \end{aligned}$$

Inserting the ansatz (19) in (17) and (18) we obtain after equating the terms of same order in ϵ and using the convention $\hat{w}_j = v_j = 0$ for $j < 0$,

$$\begin{cases} \dfrac{\partial^2}{\partial \xi^2}\hat{w}_j = -\displaystyle\sum_{k=1}^{5} A_k \hat{w}_{j-k} & \text{in} \qquad \mathcal{G}, \\ \hat{w}_j(s,0) = v_j(x_\Gamma(s)) & s \in [0, L[, \\ \dfrac{\partial \hat{w}_j}{\partial \xi}(s,0) = \dfrac{\partial v_{j-1}}{\partial \nu}(x_\Gamma(s)) & s \in [0, L[. \end{cases} \tag{20}$$

These equations can be solved inductively to get the expressions of $\hat{w}_j$ in terms of the boundary values of v_l, $l \leq j$. One gets for $j = 0, 1, 2$ and 3

$$\begin{aligned} &\hat{w}_0(s,\xi) = v_0(x_\Gamma(s)), \\ &\hat{w}_1(s,\xi) = \frac{\partial v_0}{\partial \nu}(x_\Gamma(s))\xi + v_1(x_\Gamma(s)), \\ &\hat{w}_2(s,\xi) = -\frac{\xi^2}{2}\Big(\kappa\frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) + \frac{\partial^2 w_0^-}{\partial s^2}(x_\Gamma(s)) + \lambda_0 n_1 w_0^-(x_\Gamma(s))\Big) + \frac{\partial v_1}{\partial \nu}(x_\Gamma(s)\xi + v_2(x_\Gamma(s)), \end{aligned} \tag{21}$$

and

$$\begin{aligned} \hat{w}_3(s,\xi) =& \frac{\xi^3}{6}\Big(- 2\kappa^2\frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) - 3\kappa\frac{\partial^2 w_0^-}{\partial s^2}(x_\Gamma(s)) - \kappa\lambda_0 n_1 w_0^-(x_\Gamma(s)) + \lambda_0 n_1 \frac{\partial v_0}{\partial \nu}(x_\Gamma(s))\Big) \\ &+ \frac{\xi^3}{6}\Big(\frac{\partial^3 v_0}{\partial s^2 \partial \nu}(x_\Gamma(s)) - \dot{\kappa}\frac{\partial w_0^-}{\partial s}(x_\Gamma(s))\Big) \\ &+ \frac{\xi^2}{2}\Big(\kappa\frac{\partial v_1}{\partial \nu}(x_\Gamma(s)) + \lambda_0 n_1 v_1(x_\Gamma(s)) + \lambda_1 n_1 w_0^-(x_\Gamma(s))\Big) + \frac{\partial v_2}{\partial \nu}(x_\Gamma(s))\xi + v_3(x_\Gamma(s)). \end{aligned} \tag{22}$$

Now, we also postulate the following expansion for w_ϵ^-:

$$w_\epsilon^-(x) = \sum_{j=0}^{\infty} \epsilon^j w_j^-(x) \tag{23}$$

with $w_j^- : \Omega \to \mathbb{R}$ are functions independent of ϵ. Then (w_j^-, v_j) solves

$$\begin{cases} \Delta w_j^- + \lambda_0 n_0 w_j^- = -\displaystyle\sum_{l=1}^{j} \lambda_l n_0 w_{j-l}^- & \text{in} \quad \Omega, \\ \Delta v_j + \lambda_0 v_j = -\displaystyle\sum_{l=1}^{j} \lambda_l v_{j-l} & \text{in} \quad \Omega. \end{cases} \tag{24}$$

Note that the functions w_j^- are defined in all Ω and not only Ω_ϵ^0 and therefore (23) gives a extension of w_ϵ^- to all Ω. The continuity conditions at Γ can be written as

$$\tilde{w}_\epsilon^-(s,\epsilon) = \hat{w}_\epsilon(s,1) \text{ and } \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s,\epsilon) = \frac{1}{\epsilon}\frac{\partial \hat{w}_\epsilon}{\partial \xi}(s,1)$$

where $\tilde{w}_\epsilon^-$ is defined from w_ϵ^- using the local change of variables (13) in a neighborhood of Γ. Using Taylor's expansion (up to the second order, which is sufficient to compute the first three terms in the asymptotic expansion) we get

$$\tilde{w}_\epsilon^-(s,\epsilon) = \tilde{w}_\epsilon^-(s,0) + \epsilon \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s,0) + \frac{\epsilon^2}{2}\frac{\partial^2 \tilde{w}_\epsilon^-}{\partial \eta^2}(s,0) + o(\epsilon^2) = \hat{w}_\epsilon(s,1) \tag{25}$$

and

$$\frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s,\epsilon) = \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s,0) + \epsilon \frac{\partial^2 \tilde{w}_\epsilon^-}{\partial \eta^2}(s,0) + \frac{\epsilon^2}{2}\frac{\partial^3 \tilde{w}_\epsilon^-}{\partial \eta^3}(s,0) + o(\epsilon^2) = \frac{1}{\epsilon}\frac{\partial \hat{w}_\epsilon}{\partial \xi}(s,1). \tag{26}$$

Injecting (19) and (23) into (25) and (26), we respectively obtain the following continuity conditions on Γ,

$$\begin{aligned} & w_0^-(x_\Gamma(s)) = \hat{w}_0(s,1), \\ & w_1^-(x_\Gamma(s)) + \frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) = \hat{w}_1(s,1), \\ & w_2^-(x_\Gamma(s)) + \frac{\partial w_1^-}{\partial \nu}(x_\Gamma(s)) + \frac{1}{2}\frac{\partial^2 w_0^-}{\partial \nu^2}(x_\Gamma(s)) = \hat{w}_2(s,1), \end{aligned} \tag{27}$$

and

$$\begin{aligned} & 0 = \frac{\partial \hat{w}_0}{\partial \xi}(s,1), \\ & \frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) = \frac{\partial \hat{w}_1}{\partial \xi}(s,1), \\ & \frac{\partial w_1^-}{\partial \nu}(x_\Gamma(s)) + \frac{\partial^2 w_0^-}{\partial \nu^2}(x_\Gamma(s)) = \frac{\partial \hat{w}_2}{\partial \xi}(s,1). \end{aligned} \tag{28}$$

System (24) coupled with the boundary conditions (28) and (27) provide an inductive way to determine (w_j^-, v_j). We obtain the set of equations satisfied by these terms after substituting the expressions of $\hat{w}_j(s,1)$ given by (21),(22). We hereafter summarize the set of equations obtained for (w_j^-, v_j) and how to use them to get the expressions of λ_j, $j = 0, 1, 2$.

We first obtain that the couple (w_0^-, v_0) solves

$$\begin{cases} \Delta w_0^- + \lambda_0 n_0 w_0^- = 0 & \text{in} & \Omega, \\ \Delta v_0 + \lambda_0 v_0 = 0 & \text{in} & \Omega, \\ w_0^- - v_0 = 0 & \text{on} & \Gamma, \\ \dfrac{\partial w_0^-}{\partial \nu} - \dfrac{\partial v_0}{\partial \nu} = 0 & \text{on} & \Gamma. \end{cases} \tag{29}$$

This means in particular that λ_0 is a transmission eigenvalue for the limiting problem where the thin layer is removed. We then obtain that the couple (w_1^-, v_1) satisfies

$$\left\{\begin{array}{lll} \Delta w_1^- + \lambda_0 n_0 w_1^- = -\lambda_1 n_0 w_0^- & \text{in} & \Omega, \\ \Delta v_1 + \lambda_0 v_1 = -\lambda_1 v_0 & \text{in} & \Omega, \\ w_1^- - v_1 = 0 & \text{on} & \Gamma, \\ \dfrac{\partial w_1^-}{\partial \nu} - \dfrac{\partial v_1}{\partial \nu} = \lambda_0 (n_0 - n_1) w_0^- & \text{on} & \Gamma. \end{array}\right. \tag{30}$$

Since λ_0 is an eigenvalue of the associated homogeneous system, this problem is solvable only if a compatibility condition is satisfied by the right hand sides. This compatibility condition can be obtained by multiplying the first equation with $\overline{w_0^-}$ and the second equation with $\overline{v_0}$, taking the difference then integrating by parts and using (29). One ends up with

$$\lambda_1 = \frac{\displaystyle\int_\Gamma \lambda_0 (n_0 - n_1) |w_0^-|^2 ds(x)}{\displaystyle\int_\Omega \Big(n_0 |w_0^-|^2 - |v_0|^2 \Big) dx}$$

which coincides with the expression of given in Theorem 2.1 expressed in terms of $u_0 = w_0^- - v_0$. Although not covered by the analysis of convergence, we also provide the expression of the third term in the asymptotic expression. One get that the couple (w_2^-, v_2) solves

$$\left\{\begin{array}{lll} \Delta w_2^- + \lambda_0 n_0 w_2^- = -\lambda_1 n_0 w_1^- - \lambda_2 n_0 w_0^- & \text{in} & \Omega, \\ \Delta v_2 + \lambda_0 v_2 = -\lambda_1 v_1 - \lambda_2 v_0 & \text{in} & \Omega, \\ w_2^- - v_2 = h_1 & \text{on} & \Gamma, \\ \dfrac{\partial w_2^-}{\partial \nu} - \dfrac{\partial v_2}{\partial \nu} = h_2 & \text{on} & \Gamma, \end{array}\right. \tag{31}$$

where

$$h_1 = -\frac{1}{2}\frac{\partial^2 w_0^-}{\partial \nu^2} - \frac{1}{2}\lambda_0 (n_0 - n_1) w_0^-$$

and

$$h_2 = \kappa \frac{\partial^2 w_0^-}{\partial \nu^2} - \frac{7\kappa}{2}\frac{\partial^2 w_0^-}{\partial s^2} + \Big(2\kappa^2 + \lambda_0 (n_0 + \frac{n_1}{2})\Big)\frac{\partial w_0^-}{\partial \nu} - \frac{3\kappa}{2}\frac{\partial w_0^-}{\partial s} + \frac{3}{2}\frac{\partial^3 w_0^-}{\partial \nu \partial s^2}$$
$$+ \Big(\lambda_1 (2n_1 - n_0) + \lambda_0 (\kappa (\frac{n_1}{2} - n_0))\Big) w_0^- - \frac{\partial^2 w_1^-}{\partial \nu^2} + \kappa \frac{\partial w_1^-}{\partial \nu}.$$

Writing the compatibility condition for (31), we obtain the following formula for λ_2

$$\lambda_2 \int_\Omega \frac{1}{1-n_0}\Big(\frac{1}{\lambda_0}|\Delta u_0|^2 - \lambda_0 n_0 |u_0|^2\Big) dx = -\lambda_1^2 \int_\Omega \Big(\frac{1}{\lambda_0}\Delta u_0 \bar{u}_0 + \frac{1}{1-n_0}|u_0|^2\Big) dx$$
$$- \lambda_1 \int_\Omega \frac{1}{1-n_0}\Big(u_1 \Delta \bar{u}_0 + n_0 \Delta u_1 \bar{u}_0 + 2 n_0 \lambda_0 u_1 \bar{u}_0\Big) dx$$
$$+ \int_\Gamma h_1 \frac{\partial}{\partial \nu}\Big(\frac{1}{1-n_0}(\Delta + \lambda_0)\bar{u}_0\Big) ds(x) - \int_\Gamma h_2 \Big(\frac{1}{1-n_0}(\Delta + \lambda_0)\bar{u}_0\Big) ds(x). \tag{32}$$

This complicated expression shows in particular a nonlinear dependence of λ_2 in terms of n_1. It suggests that the use of λ_2 for solutions to the inverse problem of determining n_1 may not be appropriate.

4. Convergence analysis. The main goal of this section is to prove Theorem 2.1 that provides a rigorous mathematical justification of the formal asymptotic expansion for simple real transmission eigenvalues up to the first order. The proof is split into several steps. The first one is to establish the convergence in norm of the operator $\mathcal{T}_\epsilon$ to $\mathcal{T}_0$. This ensures the convergence of λ_ϵ to λ_0. In order to get to the term of order 1 in ϵ, we shall apply the Osborn theorem which requires for instance a characterization of the pointwise asymptotic expansion of $\mathcal{T}_\epsilon(U)$ up to order 1 in ϵ (for some given function $U \in H_0^2(\Omega) \times H_0^2(\Omega)$). The latter can be obtained from the asymptotic expansions of $A_\epsilon^{-1}u$, $B_\epsilon u$ and $C_\epsilon u$ for some $u \in H_0^2(\Omega)$. The difficult part to get the expansion of $A_\epsilon^{-1}u$ since for the two others, the first order terms are vanishing. This critical result is provided by Lemma 4.5.

In all the following we use the notation

$$(f,g) := (f,g)_{H_0^2(\Omega)} = \int_\Omega \Delta f \Delta g dx \text{ and } \|g\| := (g,g)^{\frac{1}{2}}_{H_0^2(\Omega)}.$$

For an operator $A : V \to V$, $\|A\|$ denotes the operator norm. To simplify the writing, C will denote a generic constant whose value may change but remains independent from ϵ as $\epsilon \to 0$.

4.1. Pointwise convergence of the spectrum of $\mathcal{T}_\epsilon$. In this first step, we prove pointwise convergence of the spectrum of the operator $\mathcal{T}_\epsilon$ to the spectrum of $\mathcal{T}_0$. This is a direct consequence of the following convergence in norm [23, 8].

Theorem 4.1. *Assume that $n_0 \in \mathcal{C}^2(\overline{\Omega})$. Let $\mathcal{T}_\epsilon$ and $\mathcal{T}_0$ be defined by (9) and (10) respectively. Then $\mathcal{T}_\epsilon$ converges to $\mathcal{T}_0$ in the operator norm.*

Proof. The proof follows from Lemma 4.2 and Lemma 4.4 below, using the definition of $\mathcal{T}_\epsilon$ and $\mathcal{T}_0$. □

In the first lemma we prove norm convergence for B_ϵ and C_ϵ.

Lemma 4.2. *Let B_ϵ, C_ϵ, B_0 and C_0 be the operators defined by (6) and (7). Then, for sufficiently small ϵ,*

$$\|B_\epsilon - B_0\| \le C\epsilon^{\frac{1}{2}} \quad \textit{and} \quad \|C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}}\| \le C\epsilon. \tag{33}$$

Proof. From the definitions of B_ϵ and B_0, we have that for $u, \phi \in H_0^2(\Omega)$

$$\begin{aligned}((B_\epsilon - B_0)u, \phi) &= \int_\Omega \frac{1}{1-n_\epsilon}\Big(u\Delta\phi + n_\epsilon \Delta u \phi\Big)dx - \int_\Omega \frac{1}{1-n_0}\Big(u\Delta\phi + n_0\Delta u\phi\Big)dx \\ &= \int_{\Omega_\epsilon}\Big(\frac{1}{1-n_1} - \frac{1}{1-n_0}\Big)\Big(u\Delta\phi + \Delta u\phi\Big)dx.\end{aligned}$$

Therefore,

$$|((B_\epsilon - B_0)u,\phi)| \le C\Big(\|u\|_{L^\infty(\Omega)}\|\Delta\phi\|_{L^1(\Omega_\epsilon)} + \|\phi\|_{L^\infty(\Omega}\|\Delta u\|_{L^1(\Omega_\epsilon)}\Big).$$

Using the Sobolev embedding theorem and the Cauchy Schwartz inequality, we get

$$|((B_\epsilon - B_0)u,\phi)| \le C\epsilon^{\frac{1}{2}}(\|u\|_{H_0^2(\Omega)}\|\phi\|_{H_0^2(\Omega)}).$$

By choosing $\phi = (B_\epsilon - B_0)u$, we get

$$\|(B_\epsilon - B_0)u\|_{H_0^2(\Omega)} \le C\epsilon^{\frac{1}{2}}\|u\|_{H_0^2(\Omega)}.$$

The proof is similar for the second inequality. For $u, \phi \in H_0^2(\Omega)$, we have

$$((C_\epsilon - C_0)u,\phi) = \int_{\Omega_\epsilon}\Big(\frac{1}{1-n_1} - \frac{1}{1-n_0}\Big)u\phi dx \le C\Big(|\Omega_\epsilon|\|u\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)}\Big)$$

From the Sobolev embedding theorem, we obtain

$$((C_\epsilon - C_0)u,\phi) \le C\epsilon\Big(\|u\|_{H_0^2(\Omega)}\|\phi\|_{H_0^2(\Omega)}\Big).$$

By choosing $\phi = (C_\epsilon - C_0)u$, we have

$$\|(C_\epsilon - C_0)u\|_{H_0^2(\Omega)} \le C\epsilon\|u\|_{H_0^2(\Omega)}. \tag{34}$$

Using the square root Lemma in [24] and the fact that C_ϵ^n converges to C_0^n at the same order $O(\epsilon)$, we conclude that $C_\epsilon^{\frac{1}{2}}$ converges to $C_0^{\frac{1}{2}}$ at the same order $O(\epsilon)$. Hence we have

$$\|(C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})u\|_{H_0^2(\Omega)} \le C\epsilon\|u\|_{H_0^2(\Omega)}. \tag{35}$$

□

Now we show convergence in the $H_0^2(\Omega)$ norm for $A_\epsilon^{-1}f$ assuming smoothness of f. This will be useful in the proof of Lemma 4.4 since the operators B_ϵ and C_ϵ are regularizing.

LEMMA 4.3. *Let A_ϵ and A_0 be defined by (5) for $\epsilon > 0$ and $\epsilon = 0$, respectively and $f \in H_0^2(\Omega)$. If $A_0^{-1}f \in \mathcal{C}^2(\overline{\Omega})$, then for sufficiently small ϵ,*

$$\|A_\epsilon^{-1}f - A_0^{-1}f\| \leq C\epsilon^{\frac{1}{2}}. \tag{36}$$

Proof. For a fixed $f \in H_0^2(\Omega)$, define z_ϵ and z_0 in $H_0^2(\Omega)$ as $z_\epsilon = A_\epsilon^{-1}f$ and $z_0 = A_0^{-1}f$. Since $A_\epsilon z_\epsilon = A_0 z_0 = f$, we have that for $\phi \in H_0^2(\Omega)$

$$(A_\epsilon(z_\epsilon - z_0), \phi) = (A_0 z_0 - A_\epsilon z_0, \phi) = \int_{\Omega_\epsilon} \Big(\frac{1}{1-n_0} - \frac{1}{1-n_1}\Big)\Delta z_0 \Delta\phi dx. \tag{37}$$

If $z_0 \in \mathcal{C}^2(\overline{\Omega})$, we get

$$\int_{\Omega_\epsilon} \Big(\frac{1}{1-n_0} - \frac{1}{1-n_\epsilon}\Big)\Delta z_0 \Delta\phi dx \leq C\|\Delta z_0\|_\infty \int_{\Omega_\epsilon} \Delta\phi dx \leq C\epsilon^{\frac{1}{2}}\|\phi\|_{H_0^2(\Omega)}.$$

Thus, we have shown that

$$(A_\epsilon(z_\epsilon - z_0), \phi) \leq C\epsilon^{\frac{1}{2}}\|\phi\|_{H_0^2(\Omega)}.$$

By plugging in $\phi = z_\epsilon - z_0$, we obtain the desired convergence using the coercivity of A_ϵ. ∎

LEMMA 4.4. *Assume that $n_0 \in \mathcal{C}^2(\overline{\Omega})$. Let $A_\epsilon, B_\epsilon, C_\epsilon, A_0, B_0$ and C_0 be defined by (5), (6) and (7) for $\epsilon > 0$ and $\epsilon = 0$, respectively. Then for sufficiently small ϵ,*

$$\|A_\epsilon^{-1}B_\epsilon - A_0^{-1}B_0\| \underset{\epsilon\to 0}{\longrightarrow} 0 \text{ and } \|A_\epsilon^{-1}C_\epsilon^{\frac{1}{2}} - A_0^{-1}C_0^{\frac{1}{2}}\| \underset{\epsilon\to 0}{\longrightarrow} 0.$$

Proof. From (37), we have that for $f, \phi \in H_0^2(\Omega)$ and with $z_\epsilon = A_\epsilon^{-1}f$ and $z_0 = A_0^{-1}f$

$$(A_\epsilon(z_\epsilon - z_0), \phi) \leq C\|\Delta A_0^{-1}f\|_{L^2(\Omega_\epsilon)}\|\phi\|_{H_0^2(\Omega)}.$$

Furthermore,

$$\begin{aligned} \|A_\epsilon^{-1}B_\epsilon f - A_0^{-1}B_0 f\|_{H_0^2(\Omega)} &\leq \|(A_\epsilon^{-1} - A_0^{-1})B_0 f\|_{H_0^2(\Omega)} + \|A_\epsilon^{-1}(B_\epsilon - B_0)f\|_{H_0^2(\Omega)} \\ &\leq C\|\Delta A_0^{-1}B_0 f\|_{L^2(\Omega_\epsilon)} + \|A_\epsilon^{-1}\|\|(B_\epsilon - B_0)\|\|f\|_{H_0^2(\Omega)}. \end{aligned} \tag{38}$$

For estimating $\|\Delta A_0^{-1}B_0 f\|_{L^2(\Omega_\epsilon)}$, observe that $B_0 u \in H_0^2(\Omega)$ is the weak solution

$$\Delta\Delta B_0 u = \Delta\big(\frac{n_0}{1-n_0}u\big) + \frac{1}{1-n_0}\Delta u \text{ in } \Omega.$$

Classical regularity results [22, 25] and the fact that $n_0 \in \mathcal{C}^2(\overline{\Omega})$ imply that $B_0 u \in H^4(\Omega) \cap H_0^2(\Omega)$ and therefore

$$\|\Delta A_0^{-1}B_0 f\|_{H^1(\Omega)} \leq C\|f\|_{H^2(\Omega)}.$$

By the Sobolev embedding theorem, this implies that

$$\|\Delta A_0^{-1}B_0 f\|_{L^p(\Omega)} \leq C\|f\|_{H^2(\Omega)},$$

for $p > 2$. Let $\tilde{p} = \frac{p}{2} > 1$ and q such that $\dfrac{1}{\tilde{p}} + \dfrac{1}{q} = 1$.

$$\|\Delta A_0^{-1}B_0 f\|^2_{L^2(\Omega_\epsilon)} \leq \|\Delta A_0^{-1}B_0 f\|^2_{L^p(\Omega)}|\Omega_\epsilon|^{\frac{1}{q}} \leq C\epsilon^{\frac{1}{2q}}\|f\|_{H^2(\Omega)}. \tag{39}$$

From (33) we obtain

$$\|A_\epsilon^{-1}\|\|(B_\epsilon - B_0)\|\|f\|_{H_0^2(\Omega)} \leq C\epsilon^{\frac{1}{2}}\|f\|_{H_0^2(\Omega)}. \tag{40}$$

Using (38), (39) and (40) we have that

$$\|A_\epsilon^{-1}B_\epsilon - A_0^{-1}B_0\| \underset{\epsilon\to 0}{\longrightarrow} 0.$$

The second convergence result follows from similar arguments. ∎

Now we would like to obtain explicit formula for the correction term in the asymptotic expansion for the operator $\mathcal{T}_\epsilon$. More precisely, we define explicit formula for the corrector term associated with $A_\epsilon^{-1} - A_0^{-1}$.

4.2. Corrector term for $A_\epsilon^{-1} - A_0^{-1}$. In this subsection, we construct a corrector function and use it to estimate the convergence rate of $z_\epsilon = A_\epsilon^{-1}u$ for $u \in H_0^2(\Omega)$. Let $z_0 = A_0^{-1}u \in H_0^2(\Omega)$, i.e $z_0 \in H_0^2(\Omega)$ solution of

$$\Delta \frac{1}{1-n_0}\Delta z_0 = \Delta\Delta u \quad \text{in} \quad \Omega. \tag{41}$$

Inspired by the formal calculations on the previous section, namely problem (30), we define z_1 solution of

$$\begin{cases} \Delta \dfrac{1}{1-n_0}\Delta z_1 = 0 & \text{in} \quad \Omega, \\ z_1 = 0 & \text{on} \quad \Gamma, \\ \dfrac{\partial z_1}{\partial \nu} = \Big(\dfrac{1-n_1}{1-n_0} - 1\Big)\Delta z_0 & \text{on} \quad \Gamma. \end{cases} \tag{42}$$

We expect that $z_\epsilon = z_0 + \epsilon z_1 + O(\epsilon^2)$ in Ω_ϵ^0. We extend z_1 in Ω_ϵ as $\tilde{z}_1^\epsilon$ defined by

$$\tilde{z}_1^\epsilon = \begin{cases} z_1 & \text{in } \Omega_\epsilon^0, \\ z_1 - \psi & \text{in } \Omega_\epsilon \end{cases} \tag{43}$$

where ψ is a polynomial of order ≤ 3 and satisfying the boundary conditions:

$$\begin{cases} \psi = 0\ , \quad \dfrac{\partial \psi}{\partial \nu} = \Big(\dfrac{1-n_1}{1-n_0} - 1\Big)\Delta z_0 & \text{on} \quad \Gamma, \\ \psi = \dfrac{\partial \psi}{\partial \nu} = 0 & \text{on} \quad \Gamma_\epsilon. \end{cases} \tag{44}$$

This gives the following expression of ψ (that plays the role $\hat{w}_2$ in the formal calculations)

$$\psi(x) = \psi(\varphi(s,\epsilon\xi)) = \hat{\psi}(s,\xi) = \epsilon\Big(\frac{1-n_1}{1-n_0} - 1\Big)\Delta z_0(\varphi(s,0))\xi(1-\xi)^2.$$

The choice of ψ ensures in particular that $\tilde{z}_1^\epsilon \in H_0^2(\Omega)$. To simplify the notation we set

$$m := \Big(\frac{1}{1-n_0} - \frac{1}{1-n_1}\Big).$$

Now we have the following Lemma.

Lemma 4.5. *Assume that n_0 and n_1 are in $C^4(\overline{\Omega})$. Let $u \in H_0^2(\Omega)$ then set $z_\epsilon = A_\epsilon^{-1}u$ and $z_0 = A_0^{-1}u$. We define $\tilde{z}_1^\epsilon$ as in (43) and assume that $z_0 \in C^6(\overline{\Omega})$. Then we have, for sufficiently small ϵ,*

$$\|z_\epsilon - z_0 - \epsilon\tilde{z}_1^\epsilon\|_{H_0^2(\Omega)} \leq C\epsilon^{\frac{3}{2}}.$$

Proof. For any $\phi \in H_0^2(\Omega)$ we have that

$$(A_\epsilon(z_\epsilon - z_0 - \epsilon\tilde{z}_1^\epsilon), \phi) = (A_\epsilon(z_\epsilon - z_0), \phi) - \epsilon(A_\epsilon\tilde{z}_1^\epsilon, \phi). \tag{45}$$

We recall that

$$(A_\epsilon(z_\epsilon - z_0), \phi) = \int_{\Omega_\epsilon} \Big(\frac{1}{1-n_0} - \frac{1}{1-n_1}\Big)\Delta z_0 \Delta\phi dx.$$

Furthermore, we have that

$$(A_\epsilon\tilde{z}_1^\epsilon, \phi) = \int_{\Omega_\epsilon^0} \frac{1}{1-n_0}\Delta z_1 \Delta\phi dx + \int_{\Omega_\epsilon} \frac{1}{1-n_1}\Delta(z_1 - \psi)\Delta\phi dx. \tag{46}$$

Using the fact that $\Delta \dfrac{1}{1-n_0}\Delta z_1 = 0$ and the Green formula yield,

$$\begin{aligned}(A_\epsilon\tilde{z}_1^\epsilon, \phi) = &\int_{\Gamma_\epsilon} m\Delta z_1 \frac{\partial \phi}{\partial \nu} ds(x) + \int_{\Gamma_\epsilon} \Big(\frac{\partial}{\partial \nu}\Big(\frac{1}{1-n_1}\Delta z_1\Big) - \frac{\partial}{\partial \nu}\Big(\frac{1}{1-n_0}\Delta z_1\Big)\Big)\phi ds(x) \\ &+ \int_{\Gamma_\epsilon} \frac{1}{1-n_1}\Delta\psi\frac{\partial \phi}{\partial \nu} ds(x) - \int_{\Gamma_\epsilon} \frac{\partial}{\partial \nu}\Big(\frac{1}{1-n_1}\Delta\psi\Big)\phi ds(x) - \int_{\Omega_\epsilon} \Delta\frac{1}{1-n_1}\Delta\psi\phi dx.\end{aligned}$$

Using the expression of ψ we have

$$\frac{1}{1-n_1}\Delta\psi|_{\Gamma_\epsilon} = \frac{1}{1-n_1}\Delta_{s,\xi}\tilde{\psi}(s,1) = \frac{2}{\epsilon}m\Delta z_0(\varphi(s,0)),$$

$$\frac{\partial}{\partial\nu}(\frac{1}{1-n_1}\Delta\psi)|_{\Gamma_\epsilon} = \frac{\partial}{\partial\eta}(\frac{1}{1-n_1}\Delta\psi)|_{\Gamma_\epsilon} = \frac{1}{1-n_1}\frac{1}{\epsilon}\frac{\partial}{\partial\xi}(\Delta_{s,\xi}\tilde{\psi})(s,1) = \frac{6}{\epsilon^2}m\Delta z_0(\varphi(s,0)).$$

We then get after substitution of these expressions

$$\begin{aligned}(A_\epsilon\tilde{z}_1^\epsilon,\phi) &= \int_{\Gamma_\epsilon} m\Big(\Delta z_1(\varphi(s,\epsilon)) + \frac{2}{\epsilon}\Delta z_0(\varphi(s,0))\Big)\frac{\partial\phi}{\partial\nu}ds(x)\\ &- \int_{\Gamma_\epsilon} m\Big(\frac{\partial}{\partial\nu}(\Delta z_1(\varphi(s,\epsilon)\Big) + \frac{6}{\epsilon^2}\Delta z_0(\varphi(s,0)))\phi ds(x) - \int_{\Omega_\epsilon}\Delta\frac{1}{1-n_1}\Delta\psi\phi dx\\ &= \int_{\Gamma_\epsilon}\phi_1^\epsilon\frac{\partial\phi}{\partial\nu}ds(x) - \int_{\Gamma_\epsilon}\phi_2^\epsilon\phi ds(x) - \int_{\Omega_\epsilon}\Delta\frac{1}{1-n_1}\Delta\psi\phi dx\end{aligned}$$

where we have set

$$\phi_1^\epsilon(s) := m\Delta z_1(\varphi(s,\epsilon)) + \frac{2}{\epsilon}m\Delta z_0(\varphi(s,0)),$$

$$\phi_2^\epsilon(s) := m\frac{\partial}{\partial\nu}(\Delta z_1(\varphi(s,\epsilon))) + \frac{6}{\epsilon^2}m\Delta z_0(\varphi(s,0))$$

using the parametrization of the curve Γ_ϵ, $s \longmapsto \varphi(s,\epsilon)$ with φ defined by (13). Using this parametrization and setting $\tilde{\phi}(s,\eta) := \phi(\varphi(s,\eta))$ in Ω_ϵ we have

$$\int_{\Gamma_\epsilon}\phi_1^\epsilon\frac{\partial\phi}{\partial\nu}ds(x) = \int_0^L\phi_1^\epsilon\frac{\partial\tilde{\phi}}{\partial\eta}(s,\epsilon)(1+\epsilon\kappa)ds = \int_0^L\int_0^\epsilon\phi_1^\epsilon\frac{\partial^2\tilde{\phi}}{\partial\eta^2}(s,\eta)(1+\epsilon\kappa)dsd\eta.$$

From the definition of ϕ_1^ϵ we then get for $\phi\in H_0^2(\Omega)$,

$$\int_{\Gamma_\epsilon}\phi_1^\epsilon\frac{\partial\phi}{\partial\nu}ds(x) = \frac{2}{\epsilon}\int_0^L\int_0^\epsilon m\Delta z_0(\varphi(s,0))\frac{\partial^2\tilde{\phi}}{\partial\eta^2}(s,\eta)dsd\eta + O(\epsilon^{\frac{1}{2}})\|\phi\|_{H^2(\Omega)}.$$

Here and in all the following $O(\epsilon^r)$ denotes a function such that $O(\epsilon^r)\le C\epsilon^r$ for a constant C independent from the test function ϕ but that may depend on $\|z_0\|_{C^6(\overline{\Omega})}$. Using Taylor's expansion we also get for $\phi\in H_0^2(\Omega)$,

$$\begin{aligned}\int_{\Gamma_\epsilon}\phi_2^\epsilon\phi ds(x) &= \frac{\epsilon}{2}\int_0^L\int_0^\epsilon\phi_2^\epsilon\frac{\partial^2\tilde{\phi}}{\partial\eta^2}(s,\eta)(1+\epsilon\kappa)dsd\eta + O(\epsilon^{\frac{3}{2}})\|\phi\|_{H^2(\Omega)}\\ &= \frac{3}{\epsilon}\int_0^L\int_0^\epsilon m\Delta z_0(\varphi(s,0))\frac{\partial^2\tilde{\phi}}{\partial\eta^2}(s,\eta)dsd\eta + O(\epsilon^{\frac{1}{2}})\|\phi\|_{H^2(\Omega)}\end{aligned}$$

where the last equality is obtained after substituting the expression of ϕ_2^ϵ. One ends up with

$$\epsilon\int_{\Gamma_\epsilon}(\phi_1^\epsilon\frac{\partial\phi}{\partial\nu} - \phi_2^\epsilon\phi)ds(x) = -\int_0^L\int_0^\epsilon m\Delta z_0(\varphi(s,0))\frac{\partial^2\tilde{\phi}}{\partial\eta^2}(s,\eta)dsd\eta + O(\epsilon^{\frac{3}{2}})\|\phi\|_{H^2(\Omega)}.$$

Equation (45) then gives

$$\begin{aligned}&(A_\epsilon(z_\epsilon - z_0 - \epsilon\tilde{z}_1^\epsilon),\phi) = \int_{\Omega_\epsilon} m\Delta z_0\Delta\phi dx - \epsilon(A_\epsilon\tilde{z}_1^\epsilon,\phi)\\ &= \int_0^L\int_0^\epsilon m\Delta z_0(\varphi(s,\eta))\Delta\phi(\varphi(s,\eta))(1+\eta\kappa)dsd\eta - \int_0^L\int_0^\epsilon m\Delta z_0(\varphi(s,0)\frac{\partial^2\phi}{\partial\eta^2}(\varphi(s,\eta))dsd\eta\\ &- \epsilon\int_{\Omega_\epsilon}\Delta\frac{1}{1-n_1}\Delta\psi\phi dx + O(\epsilon^{\frac{3}{2}})\|\phi\|_{H^2(\Omega)}.\end{aligned}$$

We use the expression of the Laplacien in local coordinates

$$(1+\eta\kappa)\Delta\phi(\varphi(s,\eta)) = \frac{\partial}{\partial s}\Big(\frac{1}{1+\eta\kappa}\frac{\partial\tilde{\phi}}{\partial s}(s,\eta)\Big) + \kappa\frac{\partial\tilde{\phi}}{\partial\eta}(s,\eta) + (1+\eta\kappa)\frac{\partial^2\tilde{\phi}}{\partial\eta^2}(s,\eta)$$

to make the decomposition

$$\int_0^L\int_0^\epsilon m\Delta z_0(\varphi(s,\eta))\Delta\phi(\varphi(s,\eta))(1+\eta\kappa)dsd\eta = \int_0^L\int_0^\epsilon m\Delta z_0(\varphi(s,\eta))\frac{\partial}{\partial s}\Big(\frac{1}{1+\eta\kappa}\frac{\partial\tilde{\phi}}{\partial s}(s,\eta)\Big)$$
$$+\int_0^L\int_0^\epsilon m\Delta z_0(\varphi(s,\eta))\Big(\kappa\frac{\partial\tilde{\phi}}{\partial\eta}(s,\eta)+\eta\kappa\frac{\partial^2\tilde{\phi}}{\partial\eta^2}(s,\eta)\Big)+\int_0^L\int_0^\epsilon m\Delta z_0(\varphi(s,\eta))\frac{\partial^2\tilde{\phi}}{\partial\eta^2}(s,\eta).$$

To estimate the first term, we integrate by parts on $[0,L[$, we obtain

$$\int_0^L\int_0^\epsilon m\Delta z_0(\varphi(s,\eta))\frac{\partial}{\partial s}\Big(\frac{1}{1+\eta\kappa}\frac{\partial\tilde{\phi}}{\partial s}(s,\eta)\Big)d\eta ds$$
$$=-\int_0^L\int_0^\epsilon \frac{1}{1+\eta\kappa}\frac{\partial}{\partial s}(m\Delta z_0(\varphi(s,\eta)))\frac{\partial\tilde{\phi}}{\partial s}(s,\eta)dsd\eta$$
$$=-\epsilon\int_0^L\int_0^1 m\frac{1}{1+\epsilon\xi\kappa}\frac{\partial}{\partial s}\Delta z_0(\varphi(s,\epsilon\xi))\frac{\partial\tilde{\phi}}{\partial s}(s,\epsilon\xi)dsd\xi$$
$$=-\epsilon\int_0^L\int_0^1 m\frac{\partial}{\partial s}\Delta z_0(\varphi(s,0))\Big(\int_0^{\epsilon\xi}\frac{\partial^2\tilde{\phi}}{\partial\eta\partial s}(s,\eta)d\eta\Big)dsd\xi+O(\epsilon^2)\|\phi\|_{H^2(\Omega)}$$
$$=O(\epsilon^{\frac{3}{2}})\|\phi\|_{H^2(\Omega)}.$$

For the last term we proceed similarly to obtain

$$\int_0^L\int_0^\epsilon m\Delta z_0(\varphi(s,\eta))\frac{\partial\tilde{\phi}}{\partial\eta}(s,\eta)dsd\eta=\int_0^L\int_0^1 m\Delta z_0(\varphi(s,\epsilon\xi))\frac{\partial\tilde{\phi}}{\partial\eta}(s,\epsilon\xi)\epsilon dsd\xi$$
$$=\epsilon\int_0^L\int_0^1 m\Delta z_0(\varphi(s,0))\Big(\int_0^{\epsilon\xi}\frac{\partial^2\tilde{\phi}}{\partial\eta^2}(s,\eta)d\eta\Big)dsd\xi+O(\epsilon^2)\|\phi\|_{H^2(\Omega)}=O(\epsilon^{\frac{3}{2}})\|\phi\|_{H^2(\Omega)}$$

Observing in addition that

$$\int_0^L\int_0^\epsilon \eta\kappa m(\Delta z_0(\varphi(s,\eta)\frac{\partial^2\tilde{\phi}}{\partial\eta^2}(s,\eta))dsd\eta=O(\epsilon^{\frac{3}{2}})\|\phi\|_{H^2(\Omega)},$$

one ends up with

$$(A_\epsilon(z_\epsilon-z_0-\epsilon\tilde{z}_1^\epsilon),\phi)=\int_0^L\int_0^\epsilon m\Big(\Delta z_0(\varphi(s,\eta))-\Delta z_0(\varphi(s,0))\Big)\frac{\partial^2\tilde{\phi}}{\partial\eta^2}(s,\eta)dsd\eta$$
$$-\epsilon\int_{\Omega_\epsilon}\Delta\frac{1}{1-n_1}\Delta f\psi\,\phi dx+O(\epsilon^{\frac{3}{2}})\|\phi\|_{H^2(\Omega)}.$$

To conclude we just observe that the two remaining terms are also of the form $O(\epsilon^{\frac{3}{2}})\|\phi\|_{H^2(\Omega)}$. For the first term, we simply use a Taylor expansion for Δz_0 while for the second one we just use that, due to the regularity of n_0 and n_1,

$$\Delta\frac{1}{1-n_1}\Delta\psi\in L^\infty(\Omega).$$

In conclusion,

$$(A_\epsilon(z_\epsilon-z_0-\epsilon\tilde{z}_1^\epsilon),\phi)\le C\epsilon^{\frac{3}{2}}\|\phi\|_{H^2(\Omega)}.$$

Choosing $\phi=z_\epsilon-z_0-\epsilon\tilde{z}_1^\epsilon$, since the coercivity constant associated with A_ϵ is independent from ϵ, we get

$$\|z_\epsilon-z_0-\epsilon\tilde{z}_1^\epsilon\|_{H_0^2(\Omega)}\le C\epsilon^{\frac{3}{2}}$$

which ends the proof. ∎

LEMMA 4.6. *Assume that n_0 and n_1 are in $C^4(\overline{\Omega})$. If $u\in C^6(\overline{\Omega})\cap H_0^2(\Omega)$, then for sufficiently small ϵ,*

$$\|B_0(A_\epsilon^{-1}-A_0^{-1})u\|_{H_0^2(\Omega)}\le C\epsilon \text{ and } \|C_0^{\frac{1}{2}}(A_\epsilon^{-1}-A_0^{-1})u\|_{H_0^2(\Omega)}\le C\epsilon \tag{47}$$

where C independent of ϵ.

Proof. From the estimate of Lemma 4.5 we have that
$$\|z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon\|_{L^2(\Omega)} \le C\epsilon^{\frac{3}{2}}.$$
Since $\epsilon\|\tilde{z}_1^\epsilon\|_{L^2(\Omega)} = O(\epsilon)$, then
$$\|z_\epsilon - z_0\|_{L^2(\Omega)} \le C\epsilon.$$
Since B_0 is two orders smoothing, we have that
$$\|B_0(z_\epsilon - z_0)\|_{H_0^2(\Omega)} \le \|z_\epsilon - z_0\|_{L^2(\Omega)} \le C\epsilon$$
The same proof holds for $C_0^{\frac{1}{2}}$. ∎

Now to derive the eigenvalue expansion, we will apply the Theorem of Osborn [23], which we state here for reader's convenience. Suppose X is a Hilbert space with inner product denoted by $\langle , \rangle$ and $K_n : X \to X$ is a sequence of compact linear operators such that K_n converge in the operator norm to K. It then follows that the adjoint operators also converges in norm. Let μ be a nonzero eigenvalue of K of algebraic multiplicity m . It is well known that for n large enough, there exist m eigenvalues of K_n: $\mu_1^n, \mu_2^n, ..., \mu_m^n$ such that $\mu_j^n \underset{j\to\infty}{\longrightarrow} \mu$ pour tout $j = 1, ..., m$. Let E be the spectral projection onto the generalized eigenspace of K corresponding to eigenvalue μ. The space X can be decomposed in terms of the range and null space of E as $X = R(E) \oplus N(E)$. Then form the proof of Theorem 3 in [23], one can state the following theorem.

THEOREM 4.7. *Let $\phi_1, \phi_2, ..., \phi_m$ be a normalized basis for $R(E)$, and let $\phi_1^*, \phi_2^*, ..., \phi_m^*$ be the dual basis of $R(E)$ such that $\langle v, \phi_j^* \rangle = 0$ for all $v \in N(E)$. Then there exists a constant C such that :*

$$\left|\mu - \frac{1}{m}\sum_{j=1}^m \mu_j^n - \frac{1}{m}\sum_{j=1}^m \langle (K-K_n)\phi_j, \phi_j^*\rangle\right| \le C\|(K-K_n)|_{R(E)}\|\|(K^*-K_n^*)|_{R(E)^*}\| \tag{48}$$

In order to apply Theorem 4.7 and obtain explicit expression for the first order asymptotic $\sum_{j=1}^m \langle (K-K_n)\phi_j, \phi_j^*\rangle$, one has to construct the basis ϕ_j^*. Remark that in the case of selfadjoint operators, $\phi_j^* = \phi_j$, but this does not apply to our problem. One easily check that ϕ_j^* are necessarily a basis of the generalized eigenspace of K^* associated with $\overline{\mu}$.

We now turn our attention to application of this theorem with $K_n \equiv \mathcal{T}_\epsilon$ and $K \equiv \mathcal{T}_0$ and $X \equiv H_0^2(\Omega) \times H_0^2(\Omega)$. We already showed that $\mathcal{T}_\epsilon$ converges to $\mathcal{T}_0$ in the operator norm in Theorem 4.1. In order to simplify the calculations we define the inner product on $H_0^2(\Omega) \times H_0^2(\Omega)$ by:
$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix} \right\rangle := (A_0 u, w)_{H_0^2(\Omega)} + (v, z)_{H_0^2(\Omega)}.$$
Let τ_0 be a simple real eigenvalue of $\mathcal{T}_0$, then for ϵ small enough, some eigenvalue τ_ϵ of $\mathcal{T}_\epsilon$ is such that $\tau_\epsilon \underset{\epsilon\to0}{\longrightarrow} \tau_0$.

Let $U_0 = \begin{pmatrix} u_0 \\ \lambda_0 C_0^{\frac{1}{2}} u_0 \end{pmatrix}$ be an eigenvector of $\mathcal{T}_0$ associated with τ_0. Using the expression of $\mathcal{T}_0$ one easily observes that $\tilde{U}_0^* = \begin{pmatrix} u_0 \\ -\lambda_0 C_0^{\frac{1}{2}} u_0 \end{pmatrix}$ is an eigenvector of $\mathcal{T}_0^*$ associated with τ_0. Then this eigenvector is proportional to the dual basis of U_0 if and only if

$$-\beta_0 := \langle U_0, \tilde{U}_0^* \rangle = (A_0 u_0, u_0) - \lambda_0^2 (C_0 u_0, u_0) \ne 0. \tag{49}$$

We remark that since τ_0 is assumed to be a simple eigenvalue (i.e. also with geometrical multiplicity equals 1), then (49) holds. We then can define the dual vector as
$$U_0^* = \frac{-1}{\beta_0}\tilde{U}_0^*$$
and apply Theorem 4.7 to get that

$$\left|\tau_0 - \tau_\epsilon - \langle (\mathcal{T}_0 - \mathcal{T}_\epsilon)U_0, U_0^*\rangle\right| \le C\|(\mathcal{T}_0 - \mathcal{T}_\epsilon)U_0\|_{H_0^2(\Omega)}\|(\mathcal{T}_0^* - \mathcal{T}_\epsilon^*)U_0^*\|_{H_0^2(\Omega)}. \tag{50}$$

We are now in position to prove the main result of Theorem of this paper. We refer to Theorem 4.11 for an extension to the case of transmission eigenvalues with higher multiplicities.

THEOREM 4.8. *Assume that n_0 and n_1 are in $C^4(\overline{\Omega})$. Let λ_0 be a simple real transmission eigenvalue corresponding to n_0 and let $u_0 \in H_0^2(\Omega)$ be the corresponding eigenvector. This implies in particular that (49) holds. Further assume that u_0 and $A_0^{-1}u_0$ are in $C^6(\overline{\Omega})$. Then, for $\epsilon > 0$ small enough, there exists a transmission eigenvalue λ_ϵ corresponding to n_ϵ such that*

$$\frac{1}{\lambda_\epsilon} - \frac{1}{\lambda_0} = -\frac{\epsilon}{\beta_0 \lambda_0} \int_\Gamma \frac{n_0 - n_1}{(1-n_0)^2} |\Delta u_0|^2 ds(x) + O(\epsilon^{\frac{3}{2}}). \tag{51}$$

Proof. Using estimate (50) with $\lambda_0 = \dfrac{1}{\tau_0}$, we have

$$\left| \frac{1}{\lambda_0} - \frac{1}{\lambda_\epsilon} - \langle (\mathcal{T}_0 - \mathcal{T}_\epsilon) U_0, U_0^* \rangle \right| \leq C \|(\mathcal{T}_0 - \mathcal{T}_\epsilon)U_0\|_{H_0^2(\Omega)} \|(\mathcal{T}_0^* - \mathcal{T}_\epsilon^*)U_0^*\|_{H_0^2(\Omega)} \tag{52}$$

From the definition of (9) of $\mathcal{T}_\epsilon$, we have

$$\begin{aligned}
\mathcal{T}_\epsilon U_0 &= \begin{pmatrix} -A_\epsilon^{-1} B_\epsilon u_0 - \lambda_0 A_\epsilon^{-1} C_\epsilon^{\frac{1}{2}} C_0^{\frac{1}{2}} u_0 \\ C_\epsilon^{\frac{1}{2}} u_0 \end{pmatrix} \\
&= \begin{pmatrix} -A_0^{-1} B_\epsilon u_0 - \lambda_0 A_0^{-1} C_\epsilon^{\frac{1}{2}} C_0^{\frac{1}{2}} u_0 \\ C_\epsilon^{\frac{1}{2}} u_0 \end{pmatrix} + \begin{pmatrix} -(A_\epsilon^{-1} - A_0^{-1})(B_0 u_0 + \lambda_0 C_0 u_0) \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} -(A_\epsilon^{-1} - A_0^{-1})(B_\epsilon - B_0) u_0 - \lambda_0 (A_\epsilon^{-1} - A_0^{-1})(C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}}) C_0^{\frac{1}{2}} u_0 \\ 0 \end{pmatrix}
\end{aligned}$$

Using the definition (10) of $\mathcal{T}_0$, we obtain

$$\begin{aligned}
(\mathcal{T}_\epsilon - \mathcal{T}_0) U_0 &= \begin{pmatrix} -A_0^{-1}(B_\epsilon - B_0) u_0 - \lambda_0 A_0^{-1} (C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}}) C_0^{\frac{1}{2}} u_0 \\ (C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}}) u_0 \end{pmatrix} \\
&\quad + \begin{pmatrix} -(A_\epsilon^{-1} - A_0^{-1})(B_0 u_0 + \lambda_0 C_0 u_0) \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} -(A_\epsilon^{-1} - A_0^{-1})(B_\epsilon - B_0) u_0 - \lambda_0 (A_\epsilon^{-1} - A_0^{-1})(C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}}) C_0^{\frac{1}{2}} u_0 \\ 0 \end{pmatrix}
\end{aligned}$$

From (33) and (36), we have

$$\|(\mathcal{T}_\epsilon - \mathcal{T}_0) U_0\|_{H_0^2(\Omega)} = O(\epsilon^{\frac{1}{2}}). \tag{53}$$

On the other hand, we have

$$\begin{aligned}
(\mathcal{T}_\epsilon^* - \mathcal{T}_0^*) U_0^* &= -\frac{1}{\beta_0} \begin{pmatrix} -(B_\epsilon - B_0) A_0^{-1} u_0 - B_0 (A_\epsilon^{-1} - A_0^{-1}) u_0 - \lambda_0 (C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}}) C_0^{\frac{1}{2}} u_0 \\ -(C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}}) A_0^{-1} u_0 - C_0^{\frac{1}{2}} (A_\epsilon^{-1} - A_0^{-1}) u_0 \end{pmatrix} \\
&\quad - \frac{1}{\beta_0} \begin{pmatrix} -(B_\epsilon - B_0)(A_\epsilon^{-1} - A_0^{-1}) u_0 \\ -(C_\epsilon^{\frac{1}{2}} - C_0^{\frac{1}{2}})(A_\epsilon^{-1} - A_0^{-1}) u_0 \end{pmatrix}
\end{aligned}$$

From estimates (33) and (47), we obtain

$$\|(\mathcal{T}_\epsilon^* - \mathcal{T}_0^*) U_0^*\|_{H_0^2(\Omega)} = O(\epsilon). \tag{54}$$

Next, (52) implies

$$\frac{1}{\lambda_\epsilon} - \frac{1}{\lambda_0} = \langle (\mathcal{T}_\epsilon - \mathcal{T}_0) U_0, U_0^* \rangle + O(\epsilon^{\frac{3}{2}}).$$

Using the expression of U_0^* we see that

$$\begin{aligned}\langle(\mathcal{T}_\epsilon-\mathcal{T}_0)U_0,U_0^*\rangle &= \frac{1}{\beta_0}((B_\epsilon-B_0)u_0,u_0)+\frac{1}{\beta_0}\lambda_0((C_\epsilon^{\frac12}-C_0^{\frac12})C_0^{\frac12}u_0,u_0)\\ &+\frac{1}{\beta_0}(A_0((A_\epsilon^{-1}-A_0^{-1})(B_0u_0+\lambda_0C_0u_0),u_0)\\ &+\frac{1}{\beta_0}((C_\epsilon^{\frac12}-C_0^{\frac12})u_0,\lambda_0C_0^{\frac12}u_0)+O(\epsilon^{\frac32})\end{aligned}$$

Recall that, by definition of u_0

$$A_0u_0+\lambda_0B_0u_0+\lambda_0^2C_0u_0=0. \tag{55}$$

Since $C_0^{\frac12}$ is self-adjoint, we have

$$\begin{aligned}\langle(\mathcal{T}_\epsilon-\mathcal{T}_0)U_0,U_0^*\rangle &= \frac{1}{\beta_0}((B_\epsilon-B_0)u_0,u_0)+\frac{2}{\beta_0}\lambda_0((C_\epsilon^{\frac12}-C_0^{\frac12})C_0^{\frac12}u_0,u_0)\\ &-\frac{1}{\beta_0}\frac{1}{\lambda_0}((A_\epsilon^{-1}-A_0^{-1})A_0u_0,A_0u_0)+O(\epsilon^{\frac32}).\end{aligned}$$

We then deduce

$$\begin{aligned}\frac{1}{\lambda_\epsilon}-\frac{1}{\lambda_0} &= \frac{1}{\beta_0}((B_\epsilon-B_0)u_0,u_0)+\frac{2}{\beta_0}\lambda_0((C_\epsilon^{\frac12}-C_0^{\frac12})C_0^{\frac12}u_0,u_0)\\ &-\frac{1}{\beta_0}\frac{1}{\lambda_0}((A_\epsilon^{-1}-A_0^{-1})A_0u_0,A_0u_0)+O(\epsilon^{\frac32}).\end{aligned} \tag{56}$$

In order to conclude, we use the results of the two lemmas below that treat the asymptotic for each term in (56).
Applying Lemma 4.9 with $u=u_0$ and $\phi=u_0$ we have

$$((B_\epsilon-B_0)u_0,u_0)\le C\epsilon^{\frac32} \tag{57}$$

and with $u=u_0$ and $\phi=C_0^{\frac12}u_0$, we obtain

$$((C_\epsilon^{\frac12}-C_0^{\frac12})u_0,C_0^{\frac12}u_0)\le C\epsilon^2. \tag{58}$$

Applying Lemma 4.10 with $u=A_0u_0$ (therefore $z_0=u_0$) and $\phi=u_0$, we get, using the fact that A_0 is selfadjoint,

$$((A_\epsilon^{-1}-A_0^{-1})A_0u_0,A_0u_0)=\epsilon\int_0^L\frac{n_0-n_1}{(1-n_0)^2}\Delta u_0(\varphi(s,0))\Delta u_0(\varphi(s,0))ds+O(\epsilon^{\frac32})$$

We finally obtain

$$\frac{1}{\lambda_\epsilon}-\frac{1}{\lambda_0}=-\frac{\epsilon}{\beta_0\lambda_0}\int_0^L\frac{n_0-n_1}{(1-n_0)^2}|\Delta u_0(\varphi(s,0))|^2ds+O(\epsilon^{\frac32}) \tag{59}$$ □

which corresponds with the formula announced in the theorem and concludes the proof.

LEMMA 4.9. *Under the assumptions of Theorem 4.8 one has*

$$((B_\epsilon-B_0)u,\phi)\le C\epsilon^{\frac32}\|\phi\|_{H^2(\Omega)}\quad\text{and}\quad((C_\epsilon^{\frac12}-C_0^{\frac12})u,\phi)\le C\epsilon^2\|\phi\|_{H^2(\Omega)}$$

for some C independent of ϵ and ϕ.

Proof. Since

$$((B_\epsilon-B_0)u,\phi)=\int_{\Omega_\epsilon}(\frac{1}{1-n_1}-\frac{1}{1-n_0})(u\Delta\phi+\Delta u\phi)dx,$$

Using the local coordinates in Ω_ϵ, we obtain

$$\begin{aligned}\int_{\Omega_\epsilon}\Big(\frac{1}{1-n_1}-\frac{1}{1-n_0}\Big)\Delta u\phi dx &= \int_0^L\int_0^1 m\Delta u(\varphi(s,\epsilon\xi))\tilde{\phi}(s,\epsilon\xi)\epsilon(1+\xi\epsilon\kappa)dsd\xi\\ &= \int_0^L\int_0^1 m\Delta u(\varphi(s,0))\Big(\int_0^{\epsilon\xi}\frac{\partial^2\tilde{\phi}}{\partial\eta^2}(s,\eta)d\eta\Big)\epsilon(1+\xi\epsilon\kappa)dsd\xi\\ &\le C\epsilon^{\frac{3}{2}}\|\phi\|_{H^2(\Omega)}\end{aligned}$$

Hence $((B_\epsilon-B_0)u,\phi)\le C\epsilon^{\frac{3}{2}}\|\phi\|_{H^2(\Omega)}$. Similarly, we compute the asymptotic formula of

$$\begin{aligned}((C_\epsilon-C_0)u,\phi) &= \int_0^L\int_0^1\frac{n_0}{1-n_0}u(\varphi(s,\epsilon\xi))\phi(\varphi(s,\epsilon\xi))\epsilon(1+\xi\epsilon\kappa)dsd\xi\\ &= \int_0^L\int_0^1\frac{n_0}{1-n_0}\Big(\int_0^{\epsilon\xi}\frac{\partial\tilde{u}}{\partial\eta}(s,\eta)d\eta\int_0^{\epsilon\xi}\frac{\partial\tilde{\phi}}{\partial\eta}(s,\eta)d\eta\Big)\epsilon(1+\xi\epsilon\kappa)dsd\xi\\ &\le C\epsilon^2\|\phi\|_{H^2(\Omega)}\end{aligned}$$

Using the square root Lemma in [24] and the fact that C_ϵ^n converges to C_0^n at the same order $O(\epsilon^2)$, we conclude that $C_\epsilon^{\frac{1}{2}}$ converges to $C_0^{\frac{1}{2}}$ at the same order $O(\epsilon^2)$. Thus we have

$$((C_\epsilon^{\frac{1}{2}}-C_0^{\frac{1}{2}})u,\phi)\le C\epsilon^2\|\phi\|_{H^2(\Omega)}$$ ∎

Lemma 4.10. *Under the assumptions of Theorem 4.8 one has for any $\phi\in H_0^2(\Omega)\cap\mathcal{C}^4(\overline{\Omega})$*

$$(A_0(A_\epsilon^{-1}-A_0^{-1})u,\phi)=\epsilon\int_0^L\frac{n_0-n_1}{(1-n_0)^2}\Delta z_0(\varphi(s,0))\Delta\phi(\varphi(s,0))ds+O(\epsilon^{\frac{3}{2}})$$

where $z_0:=A_0^{-1}u_0$.

Proof. With $z_\epsilon:=A_\epsilon^{-1}u_0$,

$$\begin{aligned}\int_\Omega\frac{1}{1-n_0}\Delta(z_\epsilon-z_0)\Delta\phi dx &= \int_\Omega\Big(\frac{1}{1-n_0}-\frac{1}{1-n_\epsilon}\Big)\Delta z_\epsilon\Delta\phi dx=\int_{\Omega_\epsilon}m\Delta z_\epsilon\Delta\phi dx\\ &= \int_{\Omega_\epsilon}m\Delta(z_\epsilon-z_0-\epsilon\tilde{z}_1^\epsilon)\Delta\phi dx+\int_{\Omega_\epsilon}m\Delta(z_0+\epsilon\tilde{z}_1^\epsilon)\Delta\phi dx\end{aligned}$$

Applying Lemma 4.5 we obtain

$$\begin{aligned}\int_\Omega\frac{1}{1-n_0}\Delta(z_\epsilon-z_0)\Delta\phi dx &= \int_{\Omega_\epsilon}m\Delta(z_0+\epsilon\tilde{z}_1^\epsilon)\Delta\phi dx+O(\epsilon^{\frac{3}{2}})\\ &= \int_{\Omega_\epsilon}m\Delta z_0\Delta\phi dx-\epsilon\int_{\Omega_\epsilon}m\Delta\psi\Delta\phi dx+O(\epsilon^{\frac{3}{2}}).\end{aligned}$$

Making use of the local coordinates we show

$$\begin{aligned}\int_{\Omega_\epsilon}m\Delta z_0\Delta\phi dx &= \int_0^L\int_0^1 m\Delta z_0(\varphi(s,\epsilon\xi))\Delta\phi(\varphi(s,\epsilon\xi))\epsilon(1+\epsilon\xi\kappa)dsd\xi\\ &= \epsilon\int_0^L m\Delta z_0(\varphi(s,0))\Delta\phi(\varphi(s,0))ds+O(\epsilon^{\frac{3}{2}}).\end{aligned}$$

For the second term

$$\epsilon\int_{\Omega_\epsilon}m\Delta\psi\Delta\phi dx=\epsilon\int_{\Omega_\epsilon}\psi\Delta m\Delta\phi dx-\epsilon\int_\Gamma m\frac{\partial\psi}{\partial\nu}\Delta\phi ds(x)+\epsilon\int_\Gamma m\psi\frac{\partial\Delta\phi}{\partial\nu}ds(x)$$

Or $\psi|_\Gamma = 0$ and $\dfrac{\partial\psi}{\partial\nu}|_\Gamma = \Big(\dfrac{1-n_1}{1-n_0} - 1\Big)\Delta z_0(\varphi(s,0))$. Then we have

$$\begin{aligned}
\epsilon\int_{\Omega_\epsilon} m\Delta\psi\Delta\phi dx &= \epsilon\int_{\Omega_\epsilon}\psi\Delta m\Delta\phi dx - \epsilon\int_\Gamma m\frac{\partial\psi}{\partial\eta}\Delta\phi ds(x)\\
&= \epsilon\int_0^L\int_0^1\tilde\psi(s,\xi)\Delta m\Delta\phi(\varphi(s,\epsilon\xi))\epsilon(1+\epsilon\xi\kappa)dsd\xi\\
&\quad -\epsilon\int_0^L m\Big(\frac{1-n_1}{1-n_0}-1\Big)\Delta z_0(\varphi(s,0))\Delta\phi(\varphi(s,\epsilon\xi))ds\\
&= \epsilon\int_0^L m\Big(\frac{1-n_1}{1-n_0}-1\Big)\Delta z_0(\varphi(s,0))\Delta\phi(\varphi(s,0))ds + O(\epsilon^{\frac{3}{2}})
\end{aligned}$$

Consequently

$$\int_\Omega \frac{1}{1-n_0}\Delta(z_\epsilon - z_0)\Delta\phi dx = \epsilon\int_0^L \frac{n_0-n_1}{(1-n_0)^2}\Delta z_0(\varphi(s,0))\Delta\phi(\varphi(s,0))ds + O(\epsilon^{\frac{3}{2}})$$

which implies

$$(A_0(A_\epsilon^{-1} - A_0^{-1})u, \phi) = \epsilon\int_0^L \frac{n_0-n_1}{(1-n_0)^2}\Delta z_0(\varphi(s,0))\Delta\phi(\varphi(s,0))ds + O(\epsilon^{\frac{3}{2}})$$

and concludes the proof. ∎

We now indicate a possible extension to the case where the eigenvalue τ_0 is not simple. We need in that case to assume that the geometric multiplicity m coincides with the algebraic multiplicity so that a basis of $R(E)$ is formed by eigenvectors of $\mathcal{T}_0$ that we denote by $U_0^j = \begin{pmatrix} u_0^j \\ \lambda_0 C_0^{\frac{1}{2}} u_0^j\end{pmatrix}$. A basis of $R(E)^*$ is then formed by $\tilde U_0^{j*} := \begin{pmatrix} u_0^j \\ -\lambda_0 C_0^{\frac{1}{2}} u_0^j\end{pmatrix}$. If we assume that

$$-\beta_0^j := \langle U_0^j, \tilde U_0^{j*}\rangle = (A_0 u_0^j, u_0^j) - \lambda_0^2 (C_0 u_0^j, u_0^j) \neq 0, \tag{60}$$

then we can define the dual basis as

$$U_0^{j*} = \frac{-1}{\beta_0^j}\tilde U_0^{j*}.$$

Notice that the assumption on β_0^j is not guaranteed in general. Making this assumption makes the expression of the dual basis easier to express and allows us to follow the same calculations as above to express the leading term in

$$\langle(\mathcal{T}_0 - \mathcal{T}_\epsilon)U_0^j, U_0^{j*}\rangle.$$

We then obtain the following result as a consequence of the application of Theorem 4.7.

THEOREM 4.11. *Assume that n_0 and n_1 are in $C^4(\overline\Omega)$. Let λ_0 be a real transmission eigenvalue corresponding to n_0 such that the associated eigenspace is formed only with eigenvectors $u_0^j \in H_0^2(\Omega)$, $j = 1,\dots m$. Assume in addition that β_0^j defined by (60) does not vanish and that u_0^j and $A_0^{-1}u_0^j$ are in $C^6(\overline\Omega)$. Then, for $\epsilon > 0$ small enough, there exists m transmission eigenvalues λ_ϵ^j corresponding to n_ϵ such that*

$$\frac{1}{m}\sum_{i=1}^m \frac{1}{\lambda_\epsilon^j} - \frac{1}{\lambda_0} = -\frac{\epsilon}{\lambda_0}\frac{1}{m}\sum_{i=1}^m\frac{1}{\beta_0^j}\int_\Gamma \frac{n_0-n_1}{(1-n_0)^2}|\Delta u_0^j|^2 ds(x) + O(\epsilon^{\frac{3}{2}}). \tag{61}$$

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